Chapter 4 Rational Functions and Conics

Course/Section
Lesson Number
Date

Section 4.3 Partial Fractions

Section Objectives: Students will know how to recognize and find partial fraction decompositions of rational expressions.

I. Introduction (p. 350) Pace: 10 minutes

- Introduce partial fraction decomposition by stating that in Section P.5, we took two rational expressions and added or subtracted them. In this section we go backwards, once again because of calculus. In calculus, it will be important to take a function of the form $f(x)=\dfrac{3x+1}{x^2+2x-3}$ and write it in the form $f(x)=\dfrac{1}{x-1}+\dfrac{2}{x+3}$. Each fraction in this second form is a **partial fraction**, and together they make up the **partial fraction decomposition** of the first form.
- State the following **Rules for Partial Fraction Decomposition of *N*(*x*)/*D*(*x*)**, which are found on page 350 of the text and are included here for your reference.
  1. If $N(x)/D(x)$ is improper, then divide. You may need to perform the partial fraction decomposition on the (remainder)/$D(x)$.
  2. Completely factor the denominator into the product of factors of the form $(px+q)^m$ and $(ax^2+bx+c)^n$, where the quadratic factors are irreducible.
  3. For each factor of the form $(px+q)^m$, the partial fraction decomposition must include the following sum of m fractions.
  $$\frac{A_1}{(px+q)}+\frac{A_2}{(px+q)^2}+\cdots+\frac{A_m}{(px+q)^m}$$
  4. For each factor of the form $(ax^2+bx+c)^n$, the partial fraction decomposition must include the following sum of n fractions.
  $$\frac{B_1x+C_1}{(ax^2+bx+c)}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

II. Partial Fraction Decomposition (pp. 351-355) Pace: 20 minutes

Example 1. Find the partial fraction decomposition of the following.

$$\frac{3x+1}{x^2+2x-3}=\frac{3x+1}{(x-1)(x+3)}$$

$$\frac{3x+1}{(x-1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+3}$$

$$(x-1)(x+3)\frac{3x+1}{(x-1)(x+3)}=(x-1)(x+3)\left(\frac{A}{x-1}+\frac{B}{x+3}\right)$$

$$3x+1=A(x+3)+B(x-1)$$

letting $x=1$

$$4=4A\Rightarrow A=1$$

letting $x=-3$

$$-8=-4B\Rightarrow B=2$$

$$\frac{3x+1}{x^2+2x-3}=\frac{1}{x-1}+\frac{2}{x+3}$$

Tip: It should be made clear that in the third line, we multiply by the LCD. We do not just say "this numerator goes with that denominator and that numerator goes with this denominator."

Example 2. Find the partial fraction decomposition of the following.

$$\frac{5x^2+4x+6}{x^3-1}=\frac{5x^2+4x+6}{(x-1)(x^2+x+1)}$$

$$\frac{5x^2+4x+6}{(x-1)(x^2+x+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

$$(x-1)(x^2+x+1)\frac{5x^2+4x+6}{(x-1)(x^2+x+1)}=(x-1)(x^2+x+1)\left(\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}\right)$$

$$5x^2+4x+6=A(x^2+x+1)+(Bx+C)(x-1)$$

letting $x=1$

$$15=3A\Rightarrow A=5$$

letting $x=0$ and $A=5$

$$6=5-C\Rightarrow C=-1$$

letting $x=-1, C=-1$ and $A=5$

$$7=5+2B+2\Rightarrow B=0$$

$$\frac{5x^2+4x+6}{x^3-1}=\frac{5}{x-1}-\frac{1}{x^2+x+1}$$

Example 3. Find the partial fraction decomposition of the following.

$$\frac{2x+4}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}$$

$$(x+1)^2\frac{2x+4}{(x+1)^2}=(x+1)^2\left(\frac{A}{x+1}+\frac{B}{(x+1)^2}\right)$$

$$2x+4=A(x+1)+B$$

letting $x=-1$

$$2=B$$

letting $x=0$ and $B=2$

$$4=A+2\Rightarrow A=2$$

$$\frac{2x+4}{(x+1)^3}=\frac{2}{x+1}+\frac{2}{(x+1)^2}$$

Tip: You should use the procedure discussed in the *Technology* on page 351 of the text to check these examples.

Example 4. Find the partial fraction decomposition of the following.

$$\frac{x^4+x^3+2x^2+5x+5}{(x+1)(x^2+1)} = x + \frac{x^2+4x+5}{(x+1)(x^2+1)}$$

$$\frac{x^2+4x+5}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$(x+1)(x^2+1)\frac{x^2+4x+5}{(x+1)(x^2+1)} = (x+1)(x^2+1)\left(\frac{A}{x+1} + \frac{Bx+C}{x^2+1}\right)$$

$$x^2+4x+5 = A(x^2+1) + (Bx+C)(x+1)$$

letting $x = -1$

$$2 = 2A \Rightarrow A = 1$$

letting $x = 0$ and $A = 1$

$$5 = 1 + C \Rightarrow C = 4$$

letting $x = 1, C = 4,$ and $A = 1$

$$10 = 2 + 2B + 8 \Rightarrow B = 0$$